

Types of Linkage of Quadratic Pfister Forms

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Abstract

Given a field F of positive characteristic p , $\theta \in H_p^{n-1}(F)$ and $\beta, \gamma \in F^\times$, we prove that if the symbols $\theta \wedge \frac{d\beta}{\beta}$ and $\theta \wedge \frac{d\gamma}{\gamma}$ in $H_p^n(F)$ share the same factors in $H_p^1(F)$ then the symbol $\theta \wedge \frac{d\beta}{\beta} \wedge \frac{d\gamma}{\gamma}$ in $H_p^{n+1}(F)$ is trivial. We conclude that when $p = 2$, every two totally separably $(n-1)$ -linked n -fold quadratic Pfister forms are inseparably $(n-1)$ -linked. We also describe how to construct non-isomorphic n -fold Pfister forms which are totally separably (or inseparably) $(n-1)$ -linked, i.e. share all common $(n-1)$ -fold quadratic (or bilinear) Pfister factors.

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1. Introduction

Linkage of Pfister forms is a classical topic in quadratic form theory. We say that two n -fold Pfister forms over a field F are separably (inseparably, resp.) m -linked if there exists an m -fold quadratic (bilinear) Pfister form which is a common factor of both forms. When $\text{char}(F) \neq 2$, there is no difference between quadratic and bilinear factors, so the terms coincide, and we simply say m -linked.

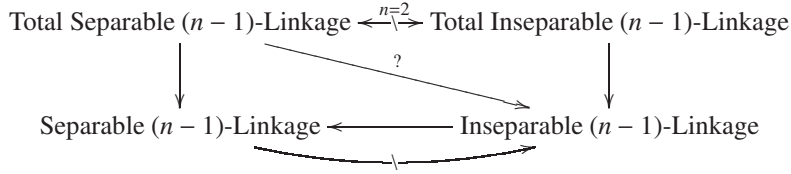
We say that two quadratic n -fold Pfister forms are totally separably (inseparably) m -linked if every quadratic (bilinear) m -fold Pfister factor of one of them is also a factor of the other. The following facts were proven in [4]:

- Two n -fold quadratic Pfister forms can be totally separably 1-linked, inseparably 1-linked, or even both, without being isometric. (The special case of quaternion algebras over fields of characteristic not 2 was covered in [9].)
- Total separable 1-linkage and total inseparable 1-linkage of n -fold quadratic Pfister forms are independent properties, i.e. do not imply each other.

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Clearly total separable (or inseparable) $(n - 1)$ -linkage implies (nontotal) separable (inseparable, resp.) $(n - 1)$ -linkage. It is known that inseparable $(n - 1)$ -linkage of quadratic n -fold Pfister forms implies separable $(n - 1)$ -linkage, but the converse is in general not true (see [12], [8], [5], [6], [3] and [1] for references).

Question 1.1. *Does total separable $(n - 1)$ -linkage of n -fold quadratic Pfister forms imply (nontotal) inseparable $(n - 1)$ -linkage?*



We answer this question in the positive in Section 3. We conclude it from deeper results on linkage of symbols in the Kato-Milne cohomology groups.

There are several other natural questions that arise in this setting:

Question 1.2. *Do there exist totally separably (inseparably) m -linked quadratic n -fold Pfister forms which are not isometric for a given $m \in \{1, \dots, n - 1\}$?*

Question 1.3. *Over fields of characteristic 2, are total separable m -linkage and total inseparable m -linkage independent properties?*

Question 1.4. *Given $1 \leq \ell < m \leq n - 1$, are there totally separably (or inseparably) ℓ -linked n -fold quadratic Pfister forms which are not totally separably (inseparably, resp.) m -linked?*

We answer Question 1.2 in full generality (in the positive), and Question 1.4 in the case of fields of characteristic not 2 and $m = n - 1$ (see Section 4). Question 1.3 was answered in the negative in [4] for $m = 1$, but it remains open for arbitrary m .

2. Preliminaries

2.1. Quadratic Forms

For general reference on symmetric bilinear forms and quadratic forms see [7]. Throughout, let F be a field and V an F -vector space. A quadratic form over F is a map $\varphi : V \rightarrow F$ such that $\varphi(av) = a^2\varphi(v)$ for all $a \in F$ and $v \in V$ and the map defined by $B_\varphi(v, w) = \varphi(v+w) - \varphi(v) - \varphi(w)$ for all $v, w \in V$ is a bilinear form on V . The bilinear form B_φ is called the polar form of φ and is clearly symmetric. Two quadratic forms $\varphi : V \rightarrow F$ and $\psi : W \rightarrow F$ are isometric if there exists an isomorphism $M : V \rightarrow W$ such that $\varphi(v) = \psi(Mv)$ for all $v \in V$. We are interested in the isometry classes of quadratic forms, so when we write $\varphi = \psi$ we actually mean that they are isometric.

We say that φ is singular if B_φ is degenerate, and that φ is nonsingular if B_φ is nondegenerate. If F is of characteristic 2, every nonsingular form φ is even dimensional and can be written as

$$\varphi = [\alpha_1, \beta_1] \perp \dots \perp [\alpha_n, \beta_n]$$

for some $\alpha_1, \dots, \beta_n \in F$, where $[\alpha, \beta]$ denotes the two-dimensional quadratic form $\psi(x, y) = \alpha x^2 + xy + \beta y^2$ and \perp denotes the orthogonal sum of quadratic forms. If the characteristic of F is different from 2, symmetric bilinear forms and quadratic forms are equivalent objects, and we do not distinguish between them in this case. The unique nonsingular two-dimensional isotropic quadratic form is $\mathbb{H} = [0, 0]$, called the hyperbolic plane. A hyperbolic form is an orthogonal sum of hyperbolic planes. Any quadratic form φ over F decomposes into an orthogonal sum of a uniquely determined anisotropic quadratic form and a number of hyperbolic planes. The number of hyperbolic planes appearing in this decomposition is called the Witt index and denoted $i_W(\varphi)$.

We denote by $\langle \alpha_1, \dots, \alpha_n \rangle$ the diagonal bilinear form given by $(x, y) \mapsto \sum_{i=1}^n \alpha_i x_i y_i$. A bilinear n -fold Pfister form over F is a symmetric bilinear form isometric to a bilinear form

$$\langle 1, \alpha_1 \rangle \otimes \langle 1, \alpha_2 \rangle \otimes \dots \otimes \langle 1, \alpha_n \rangle$$

for some $\alpha_1, \alpha_2, \dots, \alpha_n \in F^\times$. We denote such a form by $\langle\langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle\rangle$. By convention, the bilinear 0-fold Pfister form is $\langle 1 \rangle$. The 2-fold Pfister forms generate the fundamental ideal IF in the Witt ring of nondegenerate symmetric bilinear forms WF . Powers of IF are denoted by $I^n F$, and are generated by n -fold Pfister forms respectively.

Let $B : V \times V \rightarrow F$ be a symmetric bilinear form over F and $\varphi : W \rightarrow F$ be a quadratic form over F . We may define a quadratic form $B \otimes \varphi : V \otimes_F W \rightarrow F$ determined by the rule that $(B \otimes \varphi)(v \otimes w) = B(v, v) \cdot \varphi(w)$ for all $w \in W, v \in V$. We call this quadratic form the tensor product of B and φ . A quadratic n -fold Pfister form over F is a tensor product of a bilinear $(n-1)$ -fold Pfister form $\langle\langle \alpha_1, \alpha_2, \dots, \alpha_{n-1} \rangle\rangle$ and a two-dimensional quadratic form $[1, \beta]$ for some $\beta \in F$. We denote such a form by $\langle\langle \alpha_1, \dots, \alpha_{n-1}, \beta \rangle\rangle$. Quadratic n -fold Pfister forms are isotropic if and only if they are hyperbolic (see [7, (9.10)]). The 2-fold quadratic Pfister forms generate the fundamental ideal, denoted $I_q F$ or $I_q^1 F$, of the Witt group of nonsingular quadratic forms. Let $I_q^n F$ denote the subgroup generated by scalar multiples of quadratic n -fold Pfister forms.

Given a symmetric bilinear form B , we denote by $Q(B)$ the quadratic form given by the map $v \mapsto B(v, v)$.

Let π be an n -fold quadratic Pfister form over F . For $m \in \{1, \dots, n\}$, we say an m -fold quadratic (resp. bilinear) Pfister form ψ (resp. B) is a factor of π if there exists an $(n-m)$ -fold bilinear (reps. quadratic) Pfister form B' (resp. ψ') such that $\pi = B' \otimes \psi$ (resp. $\pi = B \otimes \psi'$).

Let ω be an n -fold quadratic Pfister form over F . We say π and ω are separably (resp. inseparably) m -linked if there exists an m -fold quadratic (resp. bilinear) Pfister form ψ such that ψ is a factor of both π and ω . We say π and ω are totally separably (resp. inseparably) m -linked if every quadratic (resp. bilinear) m -fold Pfister form is a factor of π if and only if it is a factor of ω . This terminology comes from the fact that in characteristic 2, the function fields of quadratic (resp. bilinear) Pfister forms are separable (resp. inseparable) extensions of the ground field.

2.2. Kato-Milne Cohomology

In this section, assume F is a field of characteristic $p > 0$. For $n > 0$, the Kato-Milne Cohomology group $H_p^{n+1}(F)$ is defined to be the cokernel of the Artin-Schreier

map

$$\begin{aligned} \wp : \Omega_F^n &\rightarrow \Omega_F^n / d\Omega_F^{n-1} \\ \alpha \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_n}{\beta_n} &\mapsto (\alpha^p - \alpha) \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_n}{\beta_n}. \end{aligned}$$

We also fix $H_p^1(F)$ to be $F/\wp(F)$. The group $\nu_F(n)$ is defined to be the kernel of this map. By [2], $\nu_F(n) \cong K_n F / pK_n F$, with the isomorphism given by

$$\frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_n}{\beta_n} \mapsto \{\beta_1, \dots, \beta_n\}.$$

It is known that $H_p^2(F) \cong {}_p Br(F)$ and for $p = 2$, $H_2^n(F) \cong I_q^n F / I_q^{n+1} F$. The first isomorphism is given by the map

$$\alpha \frac{d\beta}{\beta} \mapsto [\alpha, \beta]_{p,F}$$

where $[\alpha, \beta]_{p,F}$ stands for the symbol p -algebra

$$F\langle x, y : x^p - x = \alpha, y^p = \beta, yxy^{-1} = x + 1 \rangle.$$

The second isomorphism is given by the map

$$\alpha \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_n}{\beta_n} \mapsto \langle\langle \beta_1, \dots, \beta_n, \alpha \rangle\rangle.$$

We call the logarithmic differentials $\alpha \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_{n-1}}{\beta_{n-1}}$ in $H_p^n(F)$ and $\frac{d\gamma_1}{\gamma_1} \wedge \cdots \wedge \frac{d\gamma_m}{\gamma_m}$ in $\nu_F(m)$ “symbols”. There is a natural map

$$H_p^n(F) \times \nu_F(m) \rightarrow H_p^{m+n}(F)$$

defined by the wedge product

$$\left(\alpha \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_{n-1}}{\beta_{n-1}}, \frac{d\gamma_1}{\gamma_1} \wedge \cdots \wedge \frac{d\gamma_m}{\gamma_m} \right) \mapsto \alpha \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_{n-1}}{\beta_{n-1}} \wedge \frac{d\gamma_1}{\gamma_1} \wedge \cdots \wedge \frac{d\gamma_m}{\gamma_m}.$$

We define the linkage of symbols in an analogous manner to the linkage of Pfister forms. If a symbol ω in $H_p^{m+n}(F)$ is a wedge product $\theta \wedge \psi$ where θ is a symbol in $H_p^n(F)$ and ψ is a symbol in $\nu_F(m)$, then θ and ψ are called factors of ω . We say that two symbols π and ω are separably k -linked if they have a common factor in $H_p^k(F)$, and inseparably k -linked if they have a common factor in $\nu_F(k)$. We say that two symbols π and ω are totally separably k -linked if they share all factors in $H_p^k(F)$, and inseparably k -linked if they share all factor in $\nu_F(k)$.

3. Separably $(n - 1)$ -linked Symbols in $H_p^n(F)$

In this section, assume F is a field of characteristic $p > 0$. One of the main goals is to show that total separable $(n - 1)$ -linkage implies inseparable $(n - 1)$ -linkage for quadratic n -fold Pfister forms when $p = 2$.

Lemma 3.1. For $\alpha \in F$ and $\beta \in F^\times$, let

$$t = \alpha + \frac{(\alpha - \beta)}{\gamma}.$$

The symbol p -algebra $[\alpha, \gamma]_{p,F}$ contains the étale extension $F[x : x^p - x = t^p \gamma + \beta]$ of F .

Proof. Let i and j be a pair of generators of $[\alpha, \gamma]_{p,F}$ with $i^p - i = \alpha$, $j^p = \gamma$ and $ji j^{-1} = i + 1$. Take $x = i + t j + i j$ in $[\alpha, \gamma]_{p,F}$. Then $x^p - x$ is equal to

$$\gamma \alpha^p + \gamma^{1-p} \alpha^p - \gamma^{1-p} \beta^p + \beta = t^p \gamma + \beta$$

by [3, Lemma 3.1]. Hence the subalgebra $F[x]$ of $[\alpha, \gamma]_{p,F}$ is as required. \square

Proposition 3.2. Consider two separably $(n-1)$ -linked symbols

$$\pi = \alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \quad \text{and} \quad \omega = \alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\gamma}{\gamma}$$

in $H_p^n(F)$ and let $t = \alpha + \frac{(\alpha - \beta)}{\gamma}$. If $t^p \gamma + \beta$ is a factor in $H_p^1(F)$ of π , then the class of $\alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \wedge \frac{d\gamma}{\gamma}$ in $H_p^{n+1}(F)$ is trivial.

Proof. Note first that if $t^p \gamma + \beta = 0$ then $d\gamma \wedge d\beta = 0$ and the result holds. Assume otherwise. The class of $t^p \gamma + \beta$ in $H_p^1(F)$ is a factor of ω by Lemma 3.1, so it is a common factor of π and ω . We have

$$\alpha \frac{d\beta}{\beta} \wedge \frac{d\gamma}{\gamma} = \alpha \frac{d(\beta \gamma^{-1})}{\beta \gamma^{-1}} \wedge \frac{d(t^p \gamma + \beta)}{t^p \gamma + \beta}$$

(see [6, Lemma 5.1, (e)]). Now,

$$\begin{aligned} & \alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \wedge \frac{d\gamma}{\gamma} = \\ & \alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta \gamma^{-1}}{\beta \gamma^{-1}} \wedge \frac{d(t^p \gamma + \beta)}{t^p \gamma + \beta} = \\ & \alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \wedge \frac{d(t^p \gamma + \beta)}{t^p \gamma + \beta} - \alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\gamma}{\gamma} \wedge \frac{d(t^p \gamma + \beta)}{t^p \gamma + \beta}. \end{aligned}$$

Since $t^p \gamma + \beta$ is a factor in $H_p^1(F)$ of $\alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta}$, we have

$$\alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \wedge \frac{d(t^p \gamma + \beta)}{t^p \gamma + \beta} = (t^p \gamma + \beta) \tau \wedge \frac{d(t^p \gamma + \beta)}{t^p \gamma + \beta}$$

for some $\tau \in \nu_F(n-1)$. As $(t^p \gamma + \beta) \frac{d(t^p \gamma + \beta)}{t^p \gamma + \beta} = d(t^p \gamma + \beta)$, it is trivial in $H_p^2(F)$. Hence $\alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \wedge \frac{d(t^p \gamma + \beta)}{t^p \gamma + \beta} = 0$. Similarly, since $t^p \gamma + \beta$ is a factor in $H_p^1(F)$ of $\alpha \frac{d\gamma}{\gamma}$, we have $\alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\gamma}{\gamma} \wedge \frac{d(t^p \gamma + \beta)}{t^p \gamma + \beta} = 0$. Therefore $\alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \wedge \frac{d\gamma}{\gamma} = 0$ in $H_p^{n+1}(F)$ as required. \square

Corollary 3.3. *Let*

$$\pi = \alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \quad \text{and} \quad \omega = \alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\gamma}{\gamma}$$

be two separably $(n-1)$ -linked symbols in $H_p^n(F)$. If π and ω are totally separably 1-linked then the class of

$$\alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \wedge \frac{d\gamma}{\gamma}$$

in $H_p^{n+1}(F)$ is trivial.

When $p = 2$, by the identification of the symbols with quadratic n -fold Pfister forms, we obtain the following results:

Proposition 3.4. *Assume $p = 2$. Let*

$$\pi = \langle\langle \beta, \delta_{n-2}, \dots, \delta_1, \alpha \rangle\rangle \quad \text{and} \quad \omega = \langle\langle \gamma, \delta_{n-2}, \dots, \delta_1, \alpha \rangle\rangle$$

be two separably $(n-1)$ -linked n -fold quadratic Pfister forms over F and let $t = \alpha + \frac{(\alpha-\beta)}{\gamma}$. If the 1-fold Pfister form $\langle\langle t^p \gamma + \beta \rangle\rangle$ is a factor of π , then $\langle\langle \beta, \gamma, \delta_{n-2}, \dots, \delta_1, \alpha \rangle\rangle$ is trivial. In particular, π and ω are inseparably $(n-1)$ -linked.

Proof. By [5], the $(n+1)$ -fold Pfister form $\langle\langle \beta, \gamma, \delta_{n-1}, \dots, \delta_2, \alpha \rangle\rangle$ is trivial if and only if π and ω are inseparably $(n-1)$ -linked. \square

Corollary 3.5. *Assume $p = 2$. If a pair of separably $(n-1)$ -linked n -fold quadratic Pfister forms over F are totally separably 1-linked then they are also inseparably $(n-1)$ -linked. In particular, if a pair of n -fold quadratic Pfister forms over F are totally separably $(n-1)$ -linked then they are inseparably $(n-1)$ -linked.*

Remark 3.6. A similar result to Corollary 3.3 holds more straight-forwardly for Milnor K -groups. Let p be a prime integer, n a positive integer and F an arbitrary field. If $p = 2$ then further assume that $\sqrt{-1} \in F$. Then the following is trivial:

- If $\{\alpha\} \cup \theta \in K_n F / pK_n F$ has $\{\beta\}$ as a factor in $K_1 F / pK_1 F$ then $\{\alpha, \beta\} \cup \theta = 0$ in $K_{n+1} F / pK_{n+1} F$.
- Therefore, if $\{\alpha\} \cup \theta$ and $\{\beta\} \cup \theta$ in $K_n F / pK_n F$ are totally 1-linked then $\{\alpha, \beta\} \cup \theta = 0$ in $K_{n+1} F / pK_{n+1} F$.

Remark 3.7. The result analogous to Corollary 3.3 for inseparable linkage is also straight-forward. For fields F of characteristic $p > 0$ and positive integer n ,

- If $\omega \wedge \frac{d\beta}{\beta} \in H_p^n F$ has $\frac{d\gamma}{\gamma}$ in $v_F(1)$ as a factor then $\omega \wedge \frac{d\beta}{\beta} \wedge \frac{d\gamma}{\gamma} = 0$ in $H_p^{n+1}(F)$.
- Therefore, if $\omega \wedge \frac{d\beta}{\beta}$ and $\omega \wedge \frac{d\gamma}{\gamma}$ in $H_p^n F$ are totally inseparably 1-linked then $\omega \wedge \frac{d\beta}{\beta} \wedge \frac{d\gamma}{\gamma} = 0$ in $H_p^{n+1}(F)$.

4. Totally Linked Quadratic Pfister Forms

In [4] we considered whether total 1-linkage of Pfister forms implied isometry. In general it does not. In this section, we consider whether total m -linkage implies isometry.

Lemma 4.1. *Let n be an integer ≥ 2 and $m \in \{1, \dots, n-1\}$. Let φ be an n -fold quadratic Pfister form, π an m -fold quadratic Pfister form and θ an $(m-1)$ -fold quadratic Pfister form. Assume θ is a common factor of φ and π . Then $i_W(\varphi \perp -\pi) = 2^m$ if and only if π is a factor of φ . Otherwise, $i_W(\varphi \perp -\pi) = 2^{m-1}$.*

Proof. By [7, Corollary 24.3], $i_W(\varphi \perp -\pi)$ must be a power of 2. Since θ is a factor of φ , $i_W(\varphi \perp -\pi) \geq 2^{m-1}$. Therefore, the only other possible value is 2^m , in which case π is a subform of φ and therefore a factor of φ . \square

Lemma 4.2. *Let n be an integer ≥ 2 and $m \in \{1, \dots, n-1\}$. Let φ and ψ be two non-hyperbolic, separably $(n-1)$ -linked and totally separably m -linked n -fold quadratic Pfister forms over F . Let π be an $(m+1)$ -fold quadratic Pfister form such that π is a factor of φ but not ψ . Then there exists a field extension L such that π_L is a factor of ψ_L and φ_L and ψ_L are neither isometric nor hyperbolic.*

Proof. Since φ and ψ are separably $(n-1)$ -linked, the form $\varphi \perp \psi$ is congruent mod $I_q^{n+1}F$ to some anisotropic n -fold Pfister form ϕ . Let π_0 be an m -fold quadratic Pfister factor of π . Since φ and ψ are totally separably m -linked, π_0 is a common factor of both forms.

By Lemma 4.1, $\psi \perp -\pi$ is Witt equivalent to some anisotropic 2^n -dimensional form θ . Write $L = F(\theta)$ for the function field of θ over F . If one of the forms φ_L, ψ_L and ϕ_L were hyperbolic, then θ would be similar to a subform of the form by [7, Corollary 22.5]. However, since the forms are of the same dimension, this would imply that θ is similar to an n -fold Pfister form. This is impossible because the m th cohomological invariant of θ is nontrivial. It follows that ψ_L and φ_L are not isometric as ϕ_L is not hyperbolic. \square

Theorem 4.3. *Let n be an integer ≥ 2 and $m \in \{1, \dots, n-1\}$. Let φ and ψ be two non-hyperbolic, separably $(n-1)$ -linked and totally separably m -linked quadratic n -fold Pfister forms over F . Then there exists a field extension K of F such that φ_K and ψ_K are totally separably $(m+1)$ -linked but not isometric nor hyperbolic.*

Proof. Let S be the set of $(m+1)$ -fold quadratic Pfister forms π over F such that π is a factor of φ but not of ψ . Then as in Lemma 4.2, for each $\pi \in S$, there exists a 2^n -dimensional quadratic form θ such that θ is Witt equivalent to $\pi \perp \psi$. Let F_0 be the compositum of the function fields of all such θ .

Similarly, let T be the set of $(m+1)$ -fold quadratic Pfister forms π' over F such that π' is a factor of ψ but not of φ . Again, as in Lemma 4.2, for each $\pi' \in T$, there exists a 2^n -dimensional quadratic form θ' such that θ' is Witt equivalent to $\pi' \perp \varphi$. Let F'_0 be the field compositum of the function fields of all such θ' .

Let K_0 be the compositum of F_0 and F'_0 . Then by Lemma 4.2, φ_{K_0} and ψ_{K_0} are not isometric nor hyperbolic, and every $(m+1)$ -fold Pfister form over K_0 defined over F is

a factor of φ_{K_0} if and only if it is a factor of ψ_{K_0} . Using this construction inductively, we obtain the required field extension K/F . \square

Lemma 4.4. *Assume $\text{char}(F) = 2$. Let θ be an anisotropic n -fold bilinear Pfister form over F and let θ' be an $(n-1)$ -fold bilinear Pfister form factor of θ . Let β be an element represented by θ but not by θ' . Then $Q(\theta) = Q(\langle\langle\beta\rangle\rangle \otimes \theta')$.*

Proof. This follows easily from [7, Proposition 10.4]. \square

Lemma 4.5. *Assume $\text{char}(F) = 2$. Let n be an integer ≥ 2 and $m \in \{1, \dots, n-1\}$. Let φ be an anisotropic n -fold quadratic Pfister form, π an anisotropic m -fold bilinear Pfister form and θ an $(m-1)$ -fold bilinear Pfister form. Assume θ is a common factor of φ and π . Then $i_W(\varphi \perp Q(\pi)) = 2^m$ if and only if π is a factor of φ . Otherwise, $i_W(\varphi \perp Q(\pi)) = 2^{m-1}$.*

Proof. Since θ is a factor of φ , there exists a $2^{n-m} - 1$ dimensional bilinear form b and a quadratic 1-fold Pfister form ρ such that

$$\varphi = \theta \otimes (\langle 1 \rangle \perp b) \otimes \rho.$$

Set $\tau = \theta \otimes b \otimes \rho$ (note that $b \otimes \rho$ is the so-called pure part of the quadratic Pfister form $(\langle 1 \rangle \perp b) \otimes \rho$, see [7, p.66]). Then we have that

$$\varphi \perp Q(\theta) = 2^{m-1} \otimes \mathbb{H} \perp \tau \perp Q(\theta).$$

Note that $\tau \perp Q(\theta)$ is anisotropic as φ is anisotropic.

Suppose $\tau \perp Q(\pi)$ is isotropic. Since $\tau \perp Q(\theta)$ is anisotropic, there exists an element β represented by τ and $Q(\pi)$ but not by $Q(\theta)$. As β is represented by $Q(\pi)$ but not by $Q(\theta)$, it follows from Lemma 4.4 that $Q(\langle\langle\beta\rangle\rangle \otimes \theta) = Q(\pi)$. As β is represented by $Q(\tau)$ but not by $Q(\theta)$, it follows from [7, Proposition 15.7] that $\langle\langle\beta\rangle\rangle \otimes \theta$ is a factor of φ . In particular, φ becomes isotropic over the function field of π . Hence π is a factor of φ by [11, (1.4)] and repeated use of [7, (15.6)]. \square

Theorem 4.6. *Assume $\text{char}(F) = 2$ and n be an integer ≥ 2 and $m \in \{1, \dots, n-1\}$. Let φ and ψ be two non-hyperbolic, separably $(n-1)$ -linked and totally inseparably m -linked n -fold quadratic Pfister forms over F . Then there exists a field extension K of F such that φ_K and ψ_K are totally inseparably $(m+1)$ -linked but neither isometric nor hyperbolic.*

Proof. The result follows from Lemma 4.5 in a similar way to Theorem 4.3. \square

Using Theorems 4.3 and 4.6, one can construct examples of non-isometric pairs of n -fold quadratic Pfister forms which are totally separably (or inseparably, or both) m -linked for any $m \in \{1, \dots, n-1\}$.

Example 4.7. Start with the field $F = \mathbb{F}_2((x_1)) \dots ((x_{n+1}))$ of iterated Laurent series in $n+1$ indeterminates over \mathbb{F}_2 . The forms

$$\varphi = \langle\langle x_1, \dots, x_{n-1}, x_1 \cdot \dots \cdot x_n \rangle\rangle \quad \text{and} \quad \psi = \langle\langle x_2, \dots, x_n, x_2 \cdot \dots \cdot x_{n+1} \rangle\rangle$$

over F are $(n-1)$ -linked but not totally 1-linked (see [5, Sections 9&10]). By iterating Theorem 4.3 (or 4.6) m times, we end up with a field K over which φ_K and ψ_K are totally m -linked, but neither isometric nor hyperbolic.

Question 4.8. When $\text{char}(F) = p$, does there exist a similar process that extends two non-equal, nontrivial separably $(n - 1)$ symbols in $H_p^n(F)$ to two non-equal, nontrivial totally separably (or inseparably) $(n - 1)$ -linked symbols?

Theorem 4.9. Assume $\text{char}(F) \neq 2$ and n be an integer ≥ 3 and $m \in \{1, \dots, n - 3\}$. Let φ and ψ be two non-hyperbolic $(n - 1)$ -linked and totally m -linked n -fold quadratic Pfister forms over F . Assume there exists an $(n - 1)$ -fold quadratic Pfister form ω such that

- (a) ω is a factor of φ ,
- (b) ω is not a factor of ψ ,
- (c) there exists an $(n - 2)$ -fold quadratic Pfister form that is a factor of both ω and ψ .

Then there exists a field extension K of F such that φ_K and ψ_K are totally $(m + 1)$ -linked but not totally $(n - 1)$ -linked nor hyperbolic.

Proof. This is essentially the same proof as in Theorem 4.3. Here we just need to note the following: The form $\psi \perp -\omega$ is Witt equivalent to some anisotropic 2^n dimensional form. The latter remains anisotropic under scalar extension to L by [10, Theorem 5.4]. Therefore ω_L is not a subform of ψ_L , and that completes the proof. \square

Example 4.10. Start with the field $F = \mathbb{C}((x_1)) \dots ((x_{n+1}))$ of iterated Laurent series in $n + 1$ indeterminates over \mathbb{C} . The forms $\varphi = \langle x_1, \dots, x_n \rangle$ and $\psi = \langle x_2, \dots, x_{n+1} \rangle$ over F are $(n - 1)$ -linked but not totally 1-linked. By iterating Theorem 4.3 m times, we end up with a field K over which φ_K and ψ_K are totally m -linked, but neither hyperbolic nor isotropic. If $m \leq n - 2$ then by Theorem 4.9, φ_K and ψ_K are also not totally $(n - 1)$ -linked.

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